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A PRAGMATIC INTERPRETATION OF INTUITIONISTIC PROPOSITIONAL LOGIC*

ABSTRACT. We construct an extension $\mathscr{L}^P$ of the standard language $\mathscr{L}$ of classical propositional logic by adjoining to the alphabet of $\mathscr{L}$ a new category of logical-pragmatic signs. The well formed formulas of $\mathscr{L}$ are called *radical formulas* (**rfs**) of $\mathscr{L}^P$; **rfs** preceded by the *assertion sign* $\vdash$ constitute *elementary assertive formulas* of $\mathscr{L}^P$, which can be connected together by means of the *pragmatic connectives* N, K, A, C, E, so as to obtain the set of all the *assertive formulas* (**afs**). Every **rf** of $\mathscr{L}^P$ is endowed with a *truth value* defined classically, and every **af** is endowed with a *justification value*, defined in terms of the intuitive notion of proof and depending on the truth values of its radical subformulas. In this framework, we define the notion of *pragmatic validity* in $\mathscr{L}^P$ and yield a list of criteria of pragmatic validity which hold under the assumption that only classical metalinguistic procedures of proof be accepted. We translate the classical propositional calculus (CPC) and the intuitionistic propositional calculus (IPC) into the assertive part of $\mathscr{L}^P$ and show that this translation allows us to interpret Intuitionistic Logic as an axiomatic theory of the constructive proof concept rather than an alternative to Classical Logic. Finally, we show that our framework provides a suitable background for discussing classical problems in the philosophy of logic.

1. INTRODUCTION

We propose a pragmatic interpretation of intuitionistic logic that is based on a translation of an *intuitionistic propositional calculus* (IPC) and of a *classical propositional calculus* (CPC) into a *formalized pragmatic language* $\mathscr{L}^P$; the latter is an extension of Frege's *ideographic language*, in which the assertion sign is introduced as a constitutive part in the formulas of the logical calculus.

The purpose of our interpretation is mainly philosophical. Indeed we aim to settle the conflicts between classical and intuitionistic logic, and between the classical (correspondence) and the intuitionistic (verificationist) conceptions of truth and meaning (see Dummett, 1977, 1978, 1979, 1980; Prawitz, 1977, 1980, 1987); this will be done by introducing an *integrated* perspective which preserves both the *globality* of logic (in the sense of the *global pluralism*, which admits the existence of a plurality of mutually compatible logical systems, but not of systems which are mutually incompatible or rivals, see Haack 1978, Chapter 12) and the *classical* notion of *truth as correspondence*, which we may consider *explicated* rigorously by Tarski's semantic theory (see Tarski 1933, 1944). This goal is reached in the present paper by translating CPC and IPC into $\mathscr{L}^P$.[1] Due to the relevance of the subject, we briefly summarize here the essentials of our treatment.

Let $\mathscr{L}$ be a standard language of the classical propositional logic. We denote by $\mathscr{L}^P$ in Section 2 an extension of $\mathscr{L}$, obtained by adjoining to the logical vocabulary (alphabet) of $\mathscr{L}$ a new category of logical signs,

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that we call *logical-pragmatic* signs, which contains an *assertion* sign and *pragmatic* connectives (Definition 2.1.1). By making use of this extended vocabulary, the formation rules of $\mathscr{L}^P$ recursively define two kinds of well formed formulas in $\mathscr{L}^P$: the *radical formulas* (corresponding to the well formed formulas in $\mathscr{L}$) and the *assertive formulas*. Any assertive formula contains radical formulas as proper subformulas (Definition 2.1.2). Then the *semantic rules* of $\mathscr{L}^P$ specify the conditions that must be fulfilled, whenever a semantic interpretation of the radical formulas is given by assigning a (classical) *truth value* to every radical formula of $\mathscr{L}^P$ (Definition 2.2.1). Furthermore, the *pragmatic rules* of $\mathscr{L}^P$ specify the conditions that must be fulfilled, whenever a pragmatic evaluation of the assertive formulas is given by assigning to every assertive formula of $\mathscr{L}^P$ a *justification value* (*justified* or *unjustified*); this is defined, as the so-called intuitionistic notion of truth, in terms of the intuitive (informal) notion of *proof*, the assigment being such that the pragmatic evaluation of an assertive formula of $\mathscr{L}^P$ depends on the (semantic) assignments of truth values to its radical subformulas (Definition 2.3.1). Then, we define in Section 3 the notion of *pragmatic validity* in $\mathscr{L}^P$ by using the semantic and pragmatic rules of $\mathscr{L}^P$, and provide some (direct or indirect) criteria of validity. These are applied in Section 4 in order to explore the relations among semantic and pragmatic connectives in $\mathscr{L}^P$. The translations of CPC and IPC in $\mathscr{L}^P$ are then constructed in Section 5 in such a way that the set of all theorems of CPC bijectively corresponds (as in the original Fregean system) to the set of all *elementary* assertive formulas which are pragmatically valid in $\mathscr{L}^P$, while the set of all theorems of IPC bijectively corresponds to the set of all *complex* assertive formulas (containing only *atomic* radical formulas) which are pragmatically valid in $\mathscr{L}^P$. Finally, we discuss in Section 6 some relevant philosophical aspects of our work.

It is interesting to note that $\mathscr{L}^P$ formalizes, in particular, the analysis of all sentences in terms of *force sign* and *radical* introduced by Frege (1879, 1891, 1893, 1918) and developed by various authors, among which Reichenbach (1947, Section 57) and Stenius (1969). Yet, the Frege–Reichenbach–Stenius (FRS) model applies to elementary assertive formulas only (whose pragmatic interpretation is provided in a merely intuitive way); with this model in mind, Frege proposed his system of classical logic in terms of assertive formulas, but he could not have also given a formulation of the intuitionistic logic compatible with his system of classical logic. Our language $\mathscr{L}^P$ goes beyond the limits of the FRS model by introducing the pragmatic connectives, which allow the construction of complex assertive formulas (together with the definition of a *formal* pragmatic interpretation), and permit the translation of intuitionistic logic into $\mathscr{L}^P$.

It should also be noted that our *pragmatic* interpretation (translation) differs in two basic aspects from the *modal* interpretation (translation), proposed by Gödel (1933), McKinsey and Tarski (1948), Fitting (1969), which provides a similar solution of the conflict between intuitionistic and classical logic. Indeed, if we adopt the modal interpretation, intuitionistic logic becomes an *extended* logic (with respect to classical logic), while according to our interpretation it is a *semi-extended* logic, based on a relation of *reciprocal* extension (and/or restriction) with classical logic (see Haack 1974, Chapter 1, Section 4). Furthermore, the modal interpretation yields a rather *deviant* interpretation of intuitionistic logic with respect to the standard interpretation in terms of *proof* given by Heyting (1934, 1956) and Kreisel (1965), which can be considered as a criterion of *material adequacy* for any interpretation of intuitionistic logic. On the contrary, our interpretation recovers in a natural way the standard interpretation by means of the pragmatic notion of *justification* (see Subsections 3.1 and 5.2).

Our pragmatic interpretation of intuitionistic and classical propositional logic can be considered as a first realization, restricted to assertive sentences only, of the wider *illocutory logic* recently suggested by Searle and Vanderveken (1985). It must however be stressed that the perspective of these authors is relevantly different from our strictly logical Fregean perspective. In particular, we conceive the assertive formulas and the assertions expressed by them as purely logical entities of the formal language $\mathscr{L}^P$; moreover, the justification of an assertive formula is defined uniquely in terms of the notion of *proof*, without making reference to the speaker's intentions or beliefs, or to conditions depending on the context in which the assertion is made, as it occurs, on the contrary, in the framework forwarded by Searle and Vanderveken. Therefore, we shall disregard in this paper the (highly relevant) fact that assertions are usually thought of as "personal" acts of a specific individual, and consider assertions as completely "impersonal" acts. Thus, the assertion sign $\vdash$ which appears in the formulas of $\mathscr{L}^P$ must be intuitively intended as the impersonal performative clause "it is asserted that", or "it is assertable that", rather than the personal performative clause "I assert that". We think that our results regarding the abstract notion of assertion in the present paper can be transferred to concrete (personal) assertions whenever suitable conditions are imposed on the concrete speaker, but we will not discuss this problem here.

2. THE FORMALIZED PRAGMATIC LANGUAGE $\mathscr{L}^P$

We introduce the language $\mathscr{L}^P$ in this Section by specifying its syntactic, semantic and pragmatic structure by means of a nonformalized metalanguage, consisting of a part of the English language enriched by technical symbols (in particular, letters of the Greek alphabet having the role of metalinguistic variables).

2.1. *Syntax*

The syntax of $\mathscr{L}^P$ is specified, as usual, by providing an alphabet, that is, a set of primitive signs classified according to syntactic categories, and a (finite) set of formation rules for well formed formulas (wffs). Therefore we introduce the following definitions.

DEFINITION 2.1.1. We call *alphabet* of $\mathscr{L}^P$, and denote by $\mathscr{A}^P$, the following set of signs.

Descriptive signs: the propositional letters $p, q, r, p_1, q_1, r_1, \ldots$.

Logical-semantic signs: the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Logical-pragmatic signs: the assertion sign $\vdash$; the connectives N, K, A, C, E.

Auxiliary signs: the round brackets (,). ∎

DEFINITION 2.1.2. We call *radical* (*well formed*) *formula* (**rf**) of $\mathscr{L}^P$ every formula constructed by means of the signs in $\mathscr{A}^P$ and of the following *formation rules for radical formulas* (FRR).

FRR_1. Every propositional letter is a **rf**.

FRR_2. Let α be a **rf**; then $\neg\alpha$ is a **rf**.

FRR_3. Let α_1, α_2 be **rfs**; then $(\alpha_1 \wedge \alpha_2)$, $(\alpha_1 \vee \alpha_2)$, $(\alpha_1 \rightarrow \alpha_2)$, $(\alpha_1 \leftrightarrow \alpha_2)$ are **rfs**.

We call *assertive* (*well formed*) *formula* (**af**) of $\mathscr{L}^P$ every formula constructed by means of the following *formation rules for assertive formulas* (FRA).

FRA_1. Let α be a **rf**; then $\vdash\alpha$ is an **af**.

FRA_2. Let δ be an **af**; then $N\delta$ is an **af**.

FRA_3. Let δ_1 and δ_2 be **afs**; then $(\delta_1 K \delta_2)$, $(\delta_1 A \delta_2)$, $(\delta_1 C \delta_2)$, $(\delta_1 E \delta_2)$ are **afs**.

We denote by ψ_R and ψ_A the set of all **rfs** and **afs** respectively, and call formal language $\mathscr{L}^P$ the triple $(\mathscr{A}^P, \psi_R, \psi_A)$. In addition, we say that a **rf** is *atomic* iff it consists of a propositional letter, that an **af** is *elementary* iff it takes the form $\vdash\alpha$, with α a **rf**, and denote by ψ_R^a and ψ_A^e the sets of all atomic **rfs** and all elementary **afs** respectively. Finally, we call *molecular* **rf** any **rf** which belongs to the complement $\psi_R \backslash \psi_R^a$ of ψ_R^a in ψ_R, and *complex* **af** any **af** which belongs to the complement $\psi_A \backslash \psi_A^e$ of ψ_A^e in ψ_A. ∎

REMARK 2.1.1. First, we note that, because of the aforesaid rules, the assertion sign (that we consider a particular kind of *force*, or *pragmatic mode*, sign) neither can be iterated nor can appear within the range of a logical-semantic sign in $\mathscr{A}^P$, but it can be applied only to (atomic or molecular) **rfs**, which are considered as a whole; this feature derives from the application of a known Frege's principle in $\mathscr{L}^P$, which prohibits a force sign to be iterated, or to appear within the range of a logical-semantic sign. The reasons for these restrictions can be easily understood if one observes that the logical-semantic signs are canonically interpreted in the following as *truth functions* (Subsection 2.2), while the **afs** are not endowed with a truth value (rather, they can be endowed with a *justification value*, see Subsection 2.3). Hence the **afs** cannot be connected by means of connectives denoting truth functions (see also Reichenbach, 1947, Section 57), and the assertion sign $\vdash$ is considered as an operator that transforms **rfs** into **afs** (this exemplifies in our particular case the basic syntactic difference among force signs and signs of alethic or epistemic modality; indeed the latter denote modal operators that can be introduced in $\mathscr{L}^P$ by suitably extending it, transform **rfs** into modal **rfs**, and can be iterated or appear within the range of logical-semantic signs).

Second, we note that the set ψ_A^e of all elementary **afs** consists of all the **afs** constructed by applying the assertion sign to **rfs** in ψ_R (rule FRA_1). By using FRA_2 and FRA_3 rules we can then obtain the set $\psi_A \backslash \psi_A^e$ of all complex **afs** of $\mathscr{L}^P$, which are constructed (recursively) by means of elementary **afs** and of the logical-pragmatic signs N, K, A, C, E, introduced in Definition 2.1.1 (the symbols N, K, A, C, E, are commonly used in the Polish notation in order to denote the usual logical-semantic connectives; they will be endowed here with a different meaning, since they will be interpreted in Section 2.3 as functions whose range consists of justification values). Thus FRA_2 and FRA_3 allow us to extend the model for the pragmatic analysis of statements provided by Frege–Reichenbach–Stenius (FRS), as we have already observed in the Introduction. Indeed, the FRS model regards elementary **afs** only (according to our present terminology), since it excludes that **afs** can be logically connected, unlike **rfs**, conforming to the conventional viewpoint according to which logic only deals with formulas that can be endowed with a truth value. The pragmatic connectives introduced in $\mathscr{L}^P$ allow us to construct complex **afs**, thus overcoming the limits of the FRS model, establishing logical links between **afs** and pragmatically extending the domain of logic. We will see in Subsection 5.2 that this extension also allows us to recover intuitionistic logic within $\mathscr{L}^P$. ∎

2.2. *Semantics*

The semantic interpretation of $\mathscr{L}^P$ is introduced in a standard way by means of the following definition.

DEFINITION 2.2.1. We call *semantic interpretation* of $\mathscr{L}^P$ every pair $(\{1, 0\}, \sigma)$, where $\{1, 0\}$ is the set of *truth values* (1 standing for "true" and 0 for "false") and σ is an *assignment function*,

$$\sigma\colon \alpha \in \psi_R \rightarrowtail \sigma(\alpha) \in \{1, 0\},$$

such that the following conditions, or *truth rules* (TR), are satisfied.

TR_1. Let $\alpha \in \psi_R$; then $\sigma(\neg\alpha) = 1$ iff $\sigma(\alpha) = 0$.
TR_2. Let $\alpha_1, \alpha_2 \in \psi_R$; then

(i) $\sigma(\alpha_1 \wedge \alpha_2) = 1$ iff $\sigma(\alpha_1) = 1$ and $\sigma(\alpha_2) = 1$.
(ii) $\sigma(\alpha_1 \vee \alpha_2) = 1$ iff $\sigma(\alpha_1) = 1$ or $\sigma(\alpha_2) = 1$,
(iii) $\sigma(\alpha_1 \rightarrow \alpha_2) = 1$ iff $\sigma(\alpha_1) = 0$ or $\sigma(\alpha_2) = 1$.
(iv) $\sigma(\alpha_1 \leftrightarrow \alpha_2) = 1$ iff $\sigma(\alpha_1 \rightarrow \alpha_2) = 1$ and $\sigma(\alpha_2 \rightarrow \alpha_1) = 1$.

We denote by Σ the set of all assignment functions on ψ_R. Furthermore, we denote by $T^{\top}$ the set of all *tautologies*, and with $T^{\perp}$ the set of all *contradictions*, defined as follows.

$$T^{\top} = \{\alpha \in \psi_R \mid \forall\sigma \in \Sigma, \sigma(\alpha) = 1\},$$
$$T^{\perp} = \{\alpha \in \psi_R \mid \forall\sigma \in \Sigma, \sigma(\alpha) = 0\}.$$ ■

2.3. *Pragmatics*

The pragmatic interpretation of $\mathscr{L}^P$ is introduced by means of the following definition.

DEFINITION 2.3.1. Let $\sigma \in \Sigma$. We call *pragmatic interpretation* of $\mathscr{L}^P$ associated to σ every pair $(\{J, U\}, \pi_\sigma)$, where $\{J, U\}$ is the set of *justification values* (J standing for "justified" and U standing for "unjustified"), and π_σ is a *pragmatic evaluation function*,

$$\pi_\sigma : \delta \in \psi_A \rightarrowtail \pi_\sigma(\delta) \in \{J, U\},$$

such that the following *justification rules* (JR) and *correctness criterion* (CC) are satisfied.

JR_1. Let $\alpha \in \psi_R$; then $\pi_\sigma(\vdash\alpha) = J$ iff a proof exists that α is true, i.e. that $\sigma(\alpha) = 1$ (hence $\pi_\sigma(\vdash\alpha) = U$ iff no proof exists that α is true).

JR_2. Let $\delta \in \psi_A$; then $\pi_\sigma(N\delta) = J$ iff a proof exists that δ is unjustified, i.e., that $\pi_\sigma(\delta) = U$.

JR_3. Let $\delta_1, \delta_2 \in \psi_A$; then

(i) $\pi_\sigma(\delta_1 K \delta_2) = J$ iff $\pi_\sigma(\delta_1) = J$ and $\pi_\sigma(\delta_2) = J$;
(ii) $\pi_\sigma(\delta_1 A \delta_2) = J$ iff $\pi_\sigma(\delta_1) = J$ or $\pi_\sigma(\delta_2) = J$;
(iii) $\pi_\sigma(\delta_1 C \delta_2) = J$ iff a proof exists that $\pi_\sigma(\delta_2) = J$ whenever $\pi_\sigma(\delta_1) = J$;
(iv) $\pi_\sigma(\delta_1 E \delta_2) = J$ iff $\pi_\sigma(\delta_1 C \delta_2) = J$ and $\pi_\sigma(\delta_2 C \delta_1) = J$.

CC. Let $\alpha \in \psi_R$; then $\pi_\sigma(\vdash\alpha) = J$ implies $\sigma(\alpha) = 1$.

Finally, for every $\sigma \in \Sigma$, we denote the set of all pragmatic evaluation functions on ψ_A associated to σ by Π_σ. ■

REMARK 2.3.1. Rules JR_1–JR_3 in Definition 2.3.1 require rather extensive comments, since they recursively define the new pragmatic concept of *justified* in $\mathscr{L}^P$ by interpreting the pragmatic connectives as *justification functions*.

(i) Rules JR_1–JR_3 define the justification of any **af** (hence the concepts of *justified* and *unjustified*) in terms of a notion of proof which is left undetermined, except that it satisfies the correctness criterion CC: for every **rf** α, if there is a proof that α is true, then α is true (hence proof procedures are not arbitrary enumeration procedures for wffs). Of course, specifying a function π_σ requires that metalinguistic procedures of proof be selected; to be precise, *empirical procedures* of proof must be chosen in order to yield a justification of elementary **afs** with atomic radicals, since atomic **rfs** (propositional letters) cannot be proven logically, while *logical* procedures of proof must be introduced in order to provide a justification of elementary **afs** with molecular radicals or complex **afs**. However, in this Section we intend to introduce a purely *formal* pragmatics, in order to establish general semantic properties of the (metalinguistic) concept of justification that are independent of the specific empirical and logical procedures of proof that can be selected, so that our pragmatics can be considered *neutral* with respect to the choice between different procedures. In addition we note that our approach is also neutral with respect to the interpretation of proof as *actual* or *potential* (see Prawitz, 1980, and Loar, 1987), so that we consider the expressions "proven" and "provable", "justified" and "justifiable", "asserted" and "assertable", as equivalent, hence interchangeable in our framework.

We anticipate that the above neutrality will be partially given up in the next Section. Indeed, we will see that the criteria of pragmatic validity in Section 3 (hence the correctness and completeness theorems for our translations of classical and intuitionistic propositional logics into $\mathscr{L}^P$ in Section 5) depend on the explicit assumption that only classical logical procedures of proof be accepted for elementary **afs** with molecular radicals and for complex **afs** of $\mathscr{L}^P$. This restriction will allow us to prove, in particular, that non-classical procedures of proof are not required in order to recover intuitionistic logic into our pragmatic framework. On the contrary, we will not make any choice regarding empirical proofs, since these depend on the empirical domain on which the atomic **rfs** of $\mathscr{L}^P$ are interpreted and on the theory that describes it.

It is important to observe explicitly that the choice of the logical and empirical procedures of proof induces a selection in the set Π_σ, but it may be or may not be sufficient to pick out a single function $\pi_\sigma \in \Pi_\sigma$. For instance, whenever the atomic **rfs** of $\mathscr{L}^P$ are interpreted on an empirical domain described by Classical Physics (CP), it is apparent that the choice of standard empirical procedures of proof in CP associates a justification value (J or U) to every elementary **af** $\vdash\alpha$, with α atomic, $\vdash\alpha$ being justified iff α is true, since every empirical sentence is in principle testable in CP (if one also chooses classical logical procedures of proof, a justification value is then associated to every elementary **af** $\vdash\alpha$, again $\vdash\alpha$ being justified iff α is true). On the contrary, whenever the atomic **rfs** of $\mathscr{L}^P$ are interpreted on an empirical domain described by Quantum Physics (QP), the choice of the empirical procedures of proof that are standard in QP is not sufficient to associate a unique justification value to every elementary **af** $\vdash\alpha$ with α atomic; indeed it may happen that two atomic **rfs**, say p and q, cannot be conjointly tested in QP, so that justifying $\vdash p$ prohibits the justification of $\vdash q$ (hence $\vdash p$ justified implies $\vdash q$ unjustified) and conversely. Therefore, in the latter example many pragmatic evaluation functions can be associated to the same $\sigma \in \Sigma$, depending on the choice of the empirical sentences that we want to test (we also note that in this example an elementary **af** $\vdash\alpha$ with α atomic can be unjustified either because α is tested to be false or because no test can exist of the truth value of α, independently of the truth value of α itself).[2]

(ii) Rules JR_2, JR_3 (iii), JR_3 (iv), make reference to a notion of proof that belongs to a logical level which is higher than the logical level pertaining to the notion of proof involved in rules JR_1, JR_3 (i), JR_3 (ii). This can be better understood by considering elementary **afs** only. Indeed, let δ be an elementary **af**, that is, $\delta = \vdash\alpha$, with α an **rf**, and let σ be an assignment function; by using JR_2 we get $\pi_\sigma(N\delta) = J$ iff a proof exists that no proof exists of (the truth of) α. Analogously, let δ_1 and δ_2 be elementary **afs**, that is, $\delta_1 = \vdash\alpha_1$ and $\delta_2 = \vdash\alpha_2$, with α_1 and α_2 **rfs** whatsoever; by using JR_3 (iii) (respectively, JR_3 (iv)) we get $\pi_\sigma(\delta_1 C \delta_2) = J$ (respectively, $\pi_\sigma(\delta_1 E \delta_2) = J$) iff a proof exists that (the truth of) α_2 can be proven whenever (respectively, iff) it is possible to prove (the truth of) α_1. Thus, we see that the justification of all the **afs** considered above is defined in terms of a second-level proof, i.e., in terms of a proof regarding the existence (or the inexistence) of a proof. This feature of JR_2, JR_3 (iii), JR_3 (iv), entails two properties of our pragmatics that are relevant when translating the intuitionistic propositional logic into $\mathscr{L}^P$ (we will illustrate them in Section 4). (a) Rules JR_1–JR_3 in Definition 2.3.1 do not always allow us to evaluate the justification value of a given complex **af** of $\mathscr{L}^P$ whenever all the justification values of its elementary components are known. For instance, let $\delta \in \psi_A$; then $\pi_\sigma(\delta) = J$ implies $\pi_\sigma(N\delta) = U$, but $\pi_\sigma(\delta) = U$ does not imply that $\pi_\sigma(N\delta) = J$ (it is interesting to note that in the quantum example considered in (i), $N\vdash q$ is justified whenever $\vdash p$ is justified, even if q is true, since we have assumed that there is a proof in QP that p and q are not compatible). A similar situation occurs whenever the connective C, or E, appears in some **af**. It follows that no *justification functionality principle* holds for pragmatic connectives which is analogous to the *truth functionality principle* holding for semantic connectives (in particular, this implies that reference to the concept of proof must usually be made in our pragmatics even when evaluating the justification value of a complex **af** whose elementary components have known justification values; we briefly say that our pragmatics is not J-functional). (b) The De Morgan laws, which establish a link between the semantic connectives $\wedge$ and $\vee$, do not hold true whenever the corresponding pragmatic connectives A and K are substituted in place of $\wedge$ and $\vee$ respectively (and **afs** are considered in place of **rfs**); more generally, the pragmatic connectives, unlike semantic connectives, cannot be interdefined.

Because of the above properties, rules JR_1–JR_3 endow the pragmatic connectives with a logical behaviour which is analogous to the behaviour of the intuitionistic connectives. In addition, we note that the analogous of the modus ponens rule and of the replacement rule of classical and intuitionistic logic also hold in $\mathscr{L}^P$. Indeed, it follows from JR_3 (iii) that, whenever $\delta_1 C \delta_2$ is justified and δ_1 is justified, then δ_2 is justified; furthermore, it follows from JR_3 (iv) that, whenever $\delta_1 E \delta_2$ is justified, δ_1 can be replaced by δ_2 (and conversely) in every **af** γ without modifying the justification value of γ.

(iii) Let us focus our attention on the interpretation of J and U as *justified* and *unjustified* respectively. As illustrated by our comments in (i) and (ii), Definition 2.3.1 formalizes the properties of the metalinguistic pragmatic concepts of *justified* and *unjustified*, which are different from the metalinguistic semantic concepts of *true* and *false*. More explicitly, the assignment of a semantic interpretation on ψ_R endows every **rf** with a truth value in a classical sense; but this value can be epistemically known or unknown, and the set of all **rfs** that can be explicitly proven to be true (false), whenever a semantic interpretation is given, is a subset of all true (false) **rfs** because of the correctness criterion CC (of course, the former set may coincide with the latter under suitable assumptions, see Section 3). Now, intuitively, the **rfs** which can be explicitly proven to be true are the only **rfs** which can be justifiably asserted in $\mathscr{L}^P$ (hence, the truth value of a **rf** α is known iff either $\vdash\alpha$ or $\vdash\neg\alpha$ is justified). Accordingly, the distinction between *justified/unjustified* and *true/false* has a syntactic counterpart in our approach; indeed, truth values pertain to **rfs** only, while justification values pertain to **afs** only. A pragmatic evaluation function π_σ assigns a justification value to every **af** of $\mathscr{L}^P$, in such a way that it depends on the assignment of a truth value to its radical subformulas by means of σ. Therefore, the pragmatic notion of justification is introduced here as a metalinguistic notion which presupposes the semantic notion of truth but cannot be identified with it. Consistently, the pragmatic connectives exhibit properties which are different from the properties of the semantic connectives, as we have seen above.

(iv) One may still wonder whether the metalinguistic (pragmatic) concept of *justified* satisfies an analogue of Tarski's *T-convention* for the metalinguistic (semantic) concept of *true*, i.e., if for every $\delta \in \psi_A$ the following condition holds:

JC. "δ" is justified iff δ.

Indeed, the *T*-convention establishes, according to Tarski, a material criterion of adequacy for every definition of truth. Thus, should an analogous of the T-convention hold true for the metalinguistic notion of justification, one could suspect that a new (non classical) concept of truth has been introduced by means of this pragmatic notion. We limit ourselves here to note that the answer to such a question is negative, provided that we admit that the logical-semantic and logical-pragmatic signs of the metalanguage $\mathcal{ML}^P$ of $\mathcal{L}^P$ can be identified with those of $\mathcal{L}^P$ whenever $\mathcal{ML}^P$ is formalized, and obey the same formation rules. Indeed the expression on the left in the above condition JC is a radical formula of $\mathcal{ML}^P$, while the expression on the right is an assertive formula of $\mathcal{L}^P$. Thus, the metalinguistic logical sign "iff" that appears in JC cannot be identified with "$\leftrightarrow$" (which should connect radical formulas only) or E (which should connect assertive formulas only). It follows that JC is not an acceptable metalinguistic statement of $\mathcal{ML}^P$, which confirms that J and U are not simply a relabeling of *true* and *false*. ■

3. PRAGMATIC VALIDITY

We introduce the following definition of pragmatic validity (invalidity) in $\mathcal{L}^P$.

DEFINITION 3.1. Let $\delta \in \psi_A$. We say that δ is *pragmatically valid*, or *p-valid* (respectively, *pragmatically invalid*, or *p-invalid*) iff for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(\delta) = J$ (respectively, $\pi_\sigma(\delta) = U$). ■

Since our pragmatics is not J-functional (see Remark 2.3.1, (ii)) no general (direct) decision procedure can be given which allows the recognition of all pragmatically valid **afs**. Moreover, our above definition is still neutral with respect to the procedures of proof (see Remark 2.3.1, (i)) but the set of p-valid **afs** obviously depends on the choice of the procedures. Therefore, let us convene to adopt the following Criteria for proof procedures.

CRITERION 3.1. Let α be an **rf**. Then all classical metalinguistic procedures of proof, and only those, are accepted as logical proofs of the truth of α. ■

CRITERION 3.2. Let δ be an **af**. Then all classical metalinguistic procedures of proof, and only those, are accepted as logical proofs of the justification of δ. ■

REMARK 3.1. (i) The introduction of Criteria 3.1 and 3.2 requires some intuitive supports. Criterion 3.1 can be easily justified by noticing that it refers to the procedures of logical proof regarding (molecular) **rfs** and that it guarantees that the correctness criterion CC in Definition 2.3.1 is satisfied, all classical procedures of proof, and only those, being accepted.

But an intuitive justification of Criterion 3.2 is less simple. Indeed, this criterion refers to procedures of logical proof regarding **afs**, hence to proofs of higher order than those considered in Criterion 3.1 whenever N, or C, or E appear in the **afs** (see Remark 2.3.1, (ii)). These pragmatic connectives have a nonclassical meaning, so that one can imagine that some nonclassical procedures of proof must be introduced when dealing with complex **afs**.

In order to justify Criterion 3.2, let us consider an extension $\mathscr{L}^{P*}$ of the language $\mathscr{L}^P$, obtained by adjoining the modal operator "P" (interpreted as *proved* or *provable*, according to whether the notion of proof is interpreted as actual or potential, respectively, see Remark 2.3.1) to the logical-semantic signs in the alphabet $\mathscr{A}^P$, and the following rule to the formation rules for radical formulas in Definition 2.1.2.

FRR_4. Let α be a **rf**; then, $P\alpha$ is a **rf**.

Then, a one-to-one mapping μ of the set ψ_A of all the **afs** of $\mathscr{L}^P$ on the set ψ_R^* of all the **rfs** of $\mathscr{L}^{P*}$ is obviously induced by the following correspondence (where $\alpha, \alpha_1, \alpha_2 \in \psi_R$):

$\vdash\alpha$	$P\alpha$
$N\vdash\alpha$	$P\neg P\alpha$
$\vdash\alpha_1 K \vdash\alpha_2$	$P\alpha_1 \wedge P\alpha_2$
$\vdash\alpha_1 A \vdash\alpha_2$	$P\alpha_1 \vee P\alpha_2$
$\vdash\alpha_1 C \vdash\alpha_2$	$P(P\alpha_1 \rightarrow P\alpha_2)$
$\vdash\alpha_1 E \vdash\alpha_2$	$P(P\alpha_1 \leftrightarrow P\alpha_2)$.

It seems then natural to require that the above correspondence be such that every **af** which appears on the left side is justified if the corresponding modal **rf** which appears on the right side is true (in a suitable kripkean interpretation) and vice versa, since the modal **rfs** on the right side make explicit the justification conditions stated by the pragmatic rules for the corresponding **afs** on the left side. This requirement can be fulfilled by introducing Criterion 3.2., which makes the introduction of Criterion 3.2 intuitively acceptable.

We add that the above correspondences are analogous to the correlation rules introduced by Gödel in order to provide a modal interpretation (translation) of intuitionistic logic (see Section 1). This can be considered a further proof of the adequacy of our interpretation of intuitionistic connectives (see Subsection 5.2) in terms of pragmatic connectives. Moreover, the above correspondence shows that the modal translation of intuitionistic logic involves an ascent from one linguistic level to another, since the modal operators that appear in the **rfs** on the right side express explicitly the justification conditions established by the pragmatic rules RJ_1–RJ_3 for the corresponding assertive formulas on the left side.

(ii) Both our above criteria refer to *logical* proofs. One may then wonder whether it would be convenient to introduce some further (theory dependent) criteria for *empirical* proofs (see Remark 2.3.1, (i)) in order to specify the concept of proof completely. But we do not intend to treat empirical proofs here and do not want to state any criterion regarding them, so as to maintain the neutrality of our approach with respect to this kind of proofs and leave our scheme free of any epistemological commitment that can limit its applicability. We will only assume in the following that some kind of empirical proof is given, and hold the basic philosophical distinction between truth and provability even at the empirical level (it must be stressed that this distinction entails that the assignment of an interpretation $\sigma \in \Sigma$ such that a given atomic **rf** α is true does not imply automatically that $\vdash\alpha$ is justified). ■

By using Criterion 3.1 we can state the following proposition, which establishes a link between the justification values of elementary **afs**.

PROPOSITION 3.1. Let $\alpha \in \psi_R$ and let an assignment function $\sigma \in \Sigma$ be given. Then, either $\vdash\alpha$ or $\vdash(\neg\alpha)$ is justified (equivalently, the truth value of α is known) whenever, for every atomic **rf** p that appears in α, either $\vdash p$ or $\vdash\neg p$ is justified.

Proof. For every atomic **rf** p that appears in α, let either $\vdash$p or $\vdash\neg p$ be justified. It follows that the truth value of every p is known (Remark 2.3.1, (iii)). Therefore, a classical procedure of proof exists which yields the truth value of α, since ψ_R, endowed with the semantics introduced in Definition 2.2.1, is a classical propositional logic. Hence $\pi_\sigma(\vdash\alpha) = J$ (iff $\sigma(\alpha) = 1$) or $\pi_\sigma(\vdash(\neg\alpha)) = J$ (iff $\sigma(\alpha) = 0$) because of Criterion 3.1 and Definition 2.3.1, JR_1.

REMARK 3.2. It is important to note that our proof of Proposition 3.1 essentially depends on the fact that, if $\mathscr{L}^P$ is endowed with a semantic interpretation (see Definition 2.2.1), all moleculer formulas in ψ_R are decidable with respect to a classical concept of logical proof if all their atomic formulas are decidable. This property does not necessarily hold whenever one tries to generalize $\mathscr{L}^P$ by considering as radical formulas all the wffs of some predicate logic, instead of the wffs of a propositional logic, as we have made here. ■

By using Criteria 3.1 and 3.2 we can also state a set of relevant criteria of pragmatic validity that will be extensively employed in the following (Criterion 3.2 will be used implicitly when reasoning on the p-validity of complex **afs**.

PROPOSITION 3.2. The following criteria of pragmatic validity (PV) hold in ψ_A.

PV_1. Let $\alpha \in \Psi_R$; then $\vdash\alpha$ is p-valid (respectively, p-invalid) iff α is a tautology (respectively, a contradiction).

PV_2. Let $\delta \in \psi_A$; then $N\delta$ is p-invalid if δ is p-valid (hence δ is p-invalid if $N\delta$ is p-valid).

PV_3. Let $\delta_1, \delta_2 \in \psi_A$; then $\delta_1 K \delta_2$ is p-valid iff δ_1 and δ_2 are p-valid.

PV_4. Let $\delta_1, \delta_2 \in \psi_A$; then $\delta_1 A \delta_2$ is p-valid whenever δ_1 is p-valid or δ_2 is p-valid.

PV_5. Let $\delta_1, \delta_2 \in \psi_A$; then $\delta_1 C \delta_2$ is p-valid iff a proof exists that, for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(\delta_2) = J$ whenever $\pi_\sigma(\delta_1) = J$.

PV_6. Let $\delta_1, \delta_2 \in \psi_A$; then $\delta_1 E \delta_2$ is p-valid iff $\delta_1 C \delta_2$ is p-valid and $\delta_2 C \delta_1$ is p-valid.

PV_7. Let $\delta_1, \delta_2 \in \psi_A$ and let $\delta_1 C \delta_2$ be p-valid; then, whenever δ_1 is p-valid, also δ_2 is p-valid, and whenever δ_2 is p-invalid, also δ_1 is p-invalid.

PV_8. Let $\delta_1, \delta_2 \in \psi_A$ and let $\delta_1 E \delta_2$ be p-valid; then, δ_1 is p-valid (p-invalid) iff δ_2 is p-valid (p-invalid).

Proof. We prove the above validity criteria in sequence.

PV_1. Let $\vdash\alpha$ be p-valid; it follows from Definition 3.1 and Definition 2.3.1, JR_1, that for every $\sigma \in \Sigma$, we can yield a proof that α is true, hence $\alpha \in T^\top$. Conversely, let $\alpha \in T^\top$; hence, for every $\sigma \in \Sigma$, a classical effective procedure of proof exists (e.g., by means of truth tables) which shows that α is true. By Criterion 3.1 we conclude that using $\vdash\alpha$ is p-valid.

PV_2. Let δ be p-valid. It follows from Definition 3.1 that, for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(\delta) = J$. This means that for every $\sigma \in \Sigma$ we cannot yield a proof that δ is unjustified, hence, for every $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(N\delta) = U$. It follows that $N\delta$ is p-invalid. The statement in brackets is then obvious.

PV_3. Straightforward, because of Definition 3.1 and JR_3, (i), in Definition 2.3.1.

PV_4. Straightforward, because of Definition 3.1 and JR_3, (ii), in Definition 2.3.1.

PV_5. It follows from Definition 3.1 and JR_3 (iii) in Definition 2.3.1 that $\delta_1 C \delta_2$ is p-valid iff, for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, a proof exists that π $_\sigma(\delta_2) = J$ whenever $\pi_\sigma(\delta_1) = J$, which obviously occurs iff a proof exists that for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(\delta_2) = \mathrm{J}$ whenever $\pi_\sigma(\delta_1) = J$.

PV_6. Straightforward, because of PV_5 and JR_3 (iv), in Definition 2.3.1.

PV_7. Straightforward, because of Definition 3.1 and JR_3, (iii), in Definition 2.3.1.

PV_8. Straightforward, because of PV_6 and PV_7. ∎

REMARK 3.3. (i) Consistently with our remarks at the beginning of this section, the (direct) criteria of pragmatic validity PV_1–PV_8 do not provide a general decision procedure for all p-valid **afs** of $\mathscr{L}^P$. A general procedure, however, can be supplied by means of a (indirect) criterion of pragmatic validity that can be stated, whenever the concept of logical proof is specified by Criteria 3.1 and 3.2, by using the extension $\mathscr{L}^{P*}$ of

$\mathscr{L}^P$ and the one-to one mapping μ of ψ_A on ψ_R^* introduced in Remark 3.1 (i).

PV$_9$. Let $\delta \in \psi_A$. Then δ is p-valid iff $\mu(\delta)$ is semantically valid (of course the words "semantically valid" in PV$_9$ make reference to the criterion of semantic validity for radical formulas in a modal language with a kripkean model-theoretical semantics).

We will make reference in Subsection 5.2 to PV$_9$ in order to prove the completeness of our translation of IPC.

(ii) Let us adopt Criteria 3.1 and 3.2. We retain that the pragmatic connective C in the **af** $\delta_1 C \delta_2$ (with $\delta_1, \delta_2 \in \psi_A$) grasps the meaning of ordinary expressions such as "therefore", "then", "hence" whenever $\delta_1 C \delta_2$ is p-valid. Therefore, we introduce a relation of *logical inference* on the set ψ_A of all **afs** of $\mathscr{L}^P$ by setting:

> for every $\delta_1, \delta_2 \in \psi_A$, δ_2 can be inferred from δ_1 iff $\delta_1 C \delta_2$ is p-valid.

Our belief that the relation of logical inference in natural languages must be formalized in ψ_A takes in due account a thesis by Frege (1879; see also Martin-Löf, 1984), who interpretes the premises and conclusions of an inference process as assertions. It is apparent that this perspective differs from the viewpoint usually adopted in standard logic, where the inference relation in natural languages is formalized as a relation on the set of all propositions (or on the set of all wffs, which are the syntactic counterpart of the propositions). In our context, a relation of this kind, that we call *logical implication* in what follows, is formalized as a relation on ψ_R, defined as follows:

> for every $\alpha_1, \alpha_2 \in \psi_R$, α_1 implies α_2 iff $\alpha_1 \to \alpha_2$ is a tautology.

The above distinction between *logical inference* and *logical implication*, which is philosophically important, is possible here because of the superior expressive power of $\mathscr{L}^P$. A first limited exploration of the links between the relations of logical inference and logical implication can be done by anticipating (see Section 4, (iv)) that the following **af** (where $\alpha_1, \alpha_2 \in \psi_R$) is p-valid:

$$(\vdash(\alpha_1 \to \alpha_2))C((\vdash\alpha_1)C(\vdash\alpha_2)).$$

Indeed, this result implies, because of PV$_7$, that, if $\vdash(\alpha_1 \to \alpha_2)$ is p-valid, then $(\vdash\alpha_1)C(\vdash\alpha_2)$ is p-valid. Since $\vdash(\alpha_1 \to \alpha_2)$ is p-valid iff $(\alpha_1 \to \alpha_2)$ is a tautology, because of PV$_1$, we can write:

> α_1 implies α_2 iff $\vdash(\alpha_1 \to \alpha_2)$ is p-valid,
> $\vdash(\alpha_1 \to \alpha_2)$ is p-valid implies $(\vdash\alpha_1)C(\vdash\alpha_2)$ is p-valid,

hence, $\vdash\alpha_1$ is in the inference relation with $\vdash\alpha_2$ whenever α_1 is in the implication relation with α_2. ■

4. APPLICATIONS

By using rules JR_1–JR_3 in Definition 2.3.1 and the criteria of pragmatic validity in Proposition 3.2, we can obtain p-valid **afs**. In particular, let α, α_1, $\alpha_2 \in \psi_R$; then, the following **afs** are p-valid:

(i) $(\vdash(\neg\alpha))C(N\vdash\alpha)$;
(ii) $((\vdash\alpha_1)K(\vdash\alpha_2))E(\vdash(\alpha_1\wedge\alpha_2))$;
(iii) $((\vdash\alpha_1)A(\vdash\alpha_2))C(\vdash(\alpha_1\vee\alpha_2))$;
(iv) $(\vdash(\alpha_1\rightarrow\alpha_2))C((\vdash\alpha_1)C(\vdash\alpha_2))$;
(v) $(\vdash(\alpha_1\leftrightarrow\alpha_2))C((\vdash\alpha_1)E(\vdash\alpha_2))$.

The proof that (i)–(v) are p-valid is rather simple; we limit ourselves here to provide some hints, as follows.

The validity of (i) can be proven by using JR_2 in Definition 2.3.1 and PV_5 in Proposition 3.2.

The validity of (ii) can be proven by using JR_3 (i), in Definition 2.3.1, and PV_6 in Proposition 3.2.

The validity of (iii)–(v) can be proven by using JR_3, (ii)–(iv) respectively, in Definition 2.3.1, and PV_5 in Proposition 3.2.

The pragmatic validity of the above formulas (i)–(v) is important. Indeed, it establishes some fundamental logical links between elementary **afs** (as $\vdash(\neg\alpha)$, $\vdash(\alpha_1\wedge\alpha_2)$, etc.) and complex **afs** (as $N\vdash\alpha$, $(\vdash\alpha_1)K(\vdash\alpha_2)$, etc.), hence between semantic and pragmatic connectives.

Let us list some further p-valid **afs**, the interpretation of which is immediate:

(vi) $\vdash(\alpha\leftrightarrow\neg\neg\alpha)$;
(vii) $(\vdash\alpha)E(\vdash(\neg\neg\alpha))$;
(viii) $(\vdash\alpha)C(N(N\vdash\alpha))$;
(ix) $(N(N(N\vdash\alpha)))E(N\vdash\alpha)$.

As above, one can prove that (vi)–(ix) are p-valid by using repeatedly Definition 2.3.1 and Proposition 3.2.

The above p-valid **afs** allow us to prove some interesting properties of the connectives $\neg$ and N. Indeed, let τ be a tautology and χ be a contradiction (see Definition 2.2.1). Then $\vdash\tau$ is p-valid and $\vdash\chi$ is p-invalid because of PV_1 in Proposition 3.2. By using (i), (vii), (viii), (ix), PV_7, PV_8 and the replacement rule, we easily obtain that $N\vdash(\neg\tau)$, $N(N\vdash(\tau))$ and $N(N(N\vdash(\neg\tau)))$ are p-valid, while $N\vdash(\neg\chi)$, $N(N\vdash(\chi))$ and $N(N(N\vdash(\chi)))$ are p-invalid.

It is also important to observe that the following formulas, obtained by substituting the connective E for C in (i), (iii), (iv), (v) and (viii), are not p-valid:

(i*) $\vdash(\neg\alpha)E(N\vdash\alpha)$;
(iii*) $((\vdash\alpha_1)A(\vdash\alpha_2))E(\vdash(\alpha_1\vee\alpha_2))$;

(iv*) $((\vdash\alpha_1)C(\vdash\alpha_2))E\vdash(\alpha_1 \rightarrow \alpha_2)$;
(v*) $((\vdash\alpha_1)E(\vdash\alpha_2))E(\vdash(\alpha_1 \leftrightarrow \alpha_2))$;
(viii*) $(\vdash\alpha)E(N(N\vdash\alpha))$.

Our statement can be proven by means of suitable counterexamples. For instance, let us consider (iii*) and let us replace α_1 and α_2 with α and $\neg\alpha$ respectively. We get $((\vdash\alpha)A(\vdash\neg\alpha))E\vdash(\alpha \vee \neg\alpha))$. Now, $\alpha \vee \neg\alpha \in T^{\top}$; hence, because of PV_1 in Proposition 3.2, $\vdash(\alpha \vee \neg\alpha)$ is a p-valid **af**, while $((\vdash\alpha)A(\vdash\neg\alpha))$, which can be considered a strong version of the principle of excluded middle, is not p-valid, as it can easily be recognized by considering JR_3 (ii) in Definition 2.3.1.

A further subset of **afs** which are not p-valid can be obtained by considering pragmatic analogous of the laws of classical logic that express that semantic connectives can be interdefined, which yields the following **afs**:

(x*) $((\vdash\alpha_1)K(\vdash\alpha_2))E(N((N\vdash\alpha_1)A(N\vdash\alpha_2)))$;
(xi*) $((\vdash\alpha_1)A(\vdash\alpha_2))E(N((N\vdash\alpha_1)K(N\vdash\alpha_2)))$;
(xii) $((\vdash\alpha_1)C(\vdash\alpha_2))E((N\vdash\alpha_1)A(\vdash\alpha_2))$.

Yet, it can be proven that the following **afs** are p-valid, which show that the above laws hold in a weakened form at the pragmatic level:

(x) $((\vdash\alpha_1)K(\vdash\alpha_2))C(N((N\vdash\alpha_1)A(N\vdash\alpha_2)))$;
(xi) $((\vdash\alpha_1)A(\vdash\alpha_2))C(N((N\vdash\alpha_1)K(N\vdash\alpha_2)))$;
(xii) $((N\vdash\alpha_1)A(\vdash\alpha_2))C((\vdash\alpha_1)C(\vdash\alpha_2))$.

5. The Translations of CPC and IPC

In this Section we provide a translation in $\mathscr{L}^P$ of the version of Classical Propositional Calculus (CPC) yielded by Mendelson (1964) and by Rogers (1971), and a translation of the version of Intuitionistic Propositional Calculus (IPC) proposed by Van Dalen (1986).

5.1. *Classical Propositional Calculus*

Let us introduce a structure isomorphic to Mendelson's CPC within our pragmatically extended formalized language $\mathscr{L}^P$.

DEFINITION 5.1.1. Let $\alpha_1, \alpha_2, \alpha_3 \in \psi_R$, We call ACPC the formal calculus consisting of the set of all elementary **afs** of $\mathscr{L}^P$, endowed with the following schemes of axioms and transformation rules.

A_1. $\vdash(\alpha_1 \rightarrow (\alpha_2 \rightarrow \alpha_1))$.
A_2. $\vdash((\alpha_1 \rightarrow (\alpha_2 \rightarrow \alpha_3)) \rightarrow ((\alpha_1 \rightarrow \alpha_2) \rightarrow (\alpha_1 \rightarrow \alpha_3)))$.
A_3. $\vdash((\neg\alpha_2 \rightarrow \neg\alpha_1) \rightarrow ((\neg\alpha_2 \rightarrow \alpha_1) \rightarrow \alpha_2))$.

TFR_1. $\dfrac{\vdash\alpha_1, \vdash(\alpha_1 \to \alpha_2)}{\vdash\alpha_2}$.

TFR_2. Definitional replacement for radical formulas. ■

The following proposition relates the formal calculus ACPC and the pragmatic interpretation of $\mathscr{L}^P$ provided in Section 2.3.

PROPOSITION 5.1.1. (i) (*Correctness theorem for ACPC*). Every theorem of ACPC is a p-valid elementary assertive formula of $\mathscr{L}^P$.

(ii) (*Completeness theorem for ACPC*). Every p-valid elementary assertive formula of $\mathscr{L}^P$ is a theorem of ACPC.

Proof. Let us consider a classical propositional calculus CPC whose set of well formed formulas (wffs) reduces to the set ψ_R of all **rfs** of $\mathscr{L}^P$, whose set of (schemes of) axioms is

A_1'. $\alpha_1 \to (\alpha_2 \to \alpha_1)$
A_2'. $(\alpha_1 \to (\alpha_2 \to \alpha_3)) \to ((\alpha_1 \to \alpha_2) \to (\alpha_1 \to \alpha_3))$
A_3'. $(\neg\alpha_2 \to \neg\alpha_1) \to ((\neg\alpha_2 \to \alpha_1) \to \alpha_2)$,

whose set of inference rules reduces to the classical *modus ponens* rule, together with the definitional replacement rule. It is well known that CPC is correct and complete with respect to the semantics introduced in Definition 2.2.1, that is, every wff in CPC is a theorem iff it is a tautology.

Now, let us consider axioms A_1–A_3. It is apparent that these are assertive formulas obtained by placing the assertive sign $\vdash$ in front of A_1'–A_3', respectively. An analogous procedure tranforms the modus ponens rule of CPC into the transformation rule TFR_1 of Definition 5.1.1, while the definitional replacement rule occurs identical in both calculi. Therefore, every wff of ACPC turns out to be a theorem in ACPC iff its radical part is a theorem in CPC, that is, because of our arguments above, iff its radical part is a tautology of ψ_R. By using PV_1 in Proposition 3.2 we conclude that an **af** of ACPC is a theorem iff it is a p-valid elementary **af** of $\mathscr{L}^P$, which proves both statements (i) and (ii). ■

REMARK 5.1.1. By considering the proof of Proposition 5.1.1, we see that the propositional calculus CPC can be bijectively mapped into the calculus ACPC constructed within $\mathscr{L}^P$ in such a way that every theorem of CPC is mapped into a theorem of ACPC and conversely. Thus, we have recovered a classical propositional calculus inside our pragmatically extended classical language $\mathscr{L}^P$, and this calculus is a version of the propositional calculus originally introduced in Frege's Begriffsschrift (1879). ■

5.2. *Intuitionistic Propositional Calculus*

We introduce now a structure isomorphic to Van Dalen's IPC within our pragmatically extended formalized language $\mathscr{L}^P$.

DEFINITION 5.2.1. We call set of the *intuitionistic* **afs** of $\mathscr{L}^P$ the subset ψ_A^I of ψ_A of all **afs** of ψ_A which contain only atomic **rfs**, i.e., the set constructed by means of the following rules.

IR_1. Let $\alpha \in \psi_R^a$; then $\vdash\alpha \in \psi_A^I$.
IR_2. Let $\delta \in \psi_A^I$; then $N\delta \in \psi_A^I$.
IR_3. Let $\delta_1, \delta_2 \in \psi_A^I$; then $(\delta_1 K \delta_2)$, $(\delta_1 A \delta_2)$, $(\delta_1 C \delta_2)$, $(\delta_1 E \delta_2) \in \psi_A^I$.

Let $\delta_1, \delta_2, \delta_3 \in \psi_A^I$. We call AIPC the formal calculus consisting of the set of all intuitionistic **afs** of $\mathscr{L}^P$, endowed with the following schemes of axioms and transformation rule.

AI_1. $\delta_1 C(\delta_2 C \delta_1)$.
AI_2. $(\delta_1 C \delta_2)C((\delta_1 C(\delta_2 C \delta_3))C(\delta_1 C \delta_3))$.
AI_3. $\delta_1 C(\delta_2 C(\delta_1 K \delta_2))$.
AI_4. $(\delta_1 K \delta_2)C\delta_1$, $(\delta_1 K \delta_2)C\delta_2$.
AI_5. $\delta_1 C(\delta_1 A \delta_2)$, $\delta_2 C(\delta_1 A \delta_2)$.
AI_6. $(\delta_1 C \delta_3)C((\delta_2 C \delta_3)C((\delta_1 A \delta_2)C\delta_3))$.
AI_7. $(\delta_1 C \delta_2)C((\delta_1 C(N\delta_2))C(N\delta_1))$.
AI_8. $\delta_1 C((N\delta_1)C\delta_2)$.

TFR. $\dfrac{\delta_1,\ \delta_1 C \delta_2}{\delta_2}$. ∎

The following proposition connects the formal calculus AIPC with the pragmatic interpretation of $\mathscr{L}^P$ provided in Section 2.3.

PROPOSITION 5.2.1. (i) (*Correctness theorem for AIPC*). Every theorem of AIPC is a p-valid (intuitionistic) **af** of $\mathscr{L}^P$.

(ii) (*Completeness theorem for AIPC*). Every p-valid intuitionistic **af** of $\mathscr{L}^P$ is a theorem of AIPC.

Proof. Axioms AI_1–AI_8 can be proven to be p-valid **afs** of $\mathscr{L}^P$ by means of JR_1–JR_3 in Definition 2.3.1 and of the criteria of pragmatic validity in Proposition 3.2 (for instance, AI_1 follows from JR_3 (iii) and PV_5; AI_4 follows from JR_3 (i) and (iii), PV_3 and PV_5; AI_5 follows from JR_3 (ii) and (iii), PV_4 and PV_5) or, alternatively, by means of the indirect criterion PV_9 in Remark 3.3 (i). Furthermore, the TFR rule restates the first part of criterion PV_7 in Proposition 3.2. Therefore, statement (i) holds.

We do not possess a direct proof of statement (ii). Nevertheless, we can provide an indirect proof of it by using the PV_9 criterion in Remark 3.3. Indeed, by restricting to ψ_A^I the bijective mapping μ introduced in Remark 3.2, we obtain that AIPC corresponds to a modal system of type

S4, in the sense that every theorem of AIPC corresponds to a theorem in S4 and vice versa. Now, S4 has been proven to be complete (see Hughes–Cresswell, 1968; Chellas, 1980) with respect to all kripkean interpretations with *reflexive and transitive accessibility relation*, hence with respect to the notion of semantic validity in $\mathscr{L}^{P*}$ (see Remark 3.3 (i)). This implies, because of PV_9, that AIPC is complete with respect to the pragmatic notion of validity. ■

REMARK 5.2.1. (i) Axioms AI_1–AI_8 of AIPC contain all pragmatic connectives and not only two (semantic) connectives as axioms A_1–A_3 of ACPC (the choice of $\neg$ and $\rightarrow$ in A_1–A_3 is obviously arbitrary; by analogy with CPC, several calculi can be constructed which are logically equivalent to ACPC and whose axioms only contain the connectives $\neg$ and $\wedge$, or $\neg$ and $\vee$). This is due to the fact that the interdefinition laws, which hold for semantic (classical) connectives, do not yield p-valid **afs** whenever the corresponding pragmatic (intuitionistic) connectives are substituted for the semantic ones (see Remark 2.3.1, (ii), and Section 4). Therefore, K, A, C and E cannot be defined by using one of them, together with the negation N, as it occurs in the case of the corresponding semantic connectives.

We also note that axioms AI_1–AI_8 and rule TFR have been chosen so as to restate Van Dalen's IPC by means of our pragmatic formalized language $\mathscr{L}^P$. Hence, every intuitionistic propositional calculus admits a (strong, bijective) canonical translation onto the formal calculus AIPC, which is such that every propositional letter is mapped onto an elementary **af** in ψ_A^1, every connective onto a corresponding pragmatic connective in AIPC, every theorem onto a theorem of AIPC. Thus, we say that we have recovered an intuitionistic propositional calculus into our pragmatically extended formal language $\mathscr{L}^P$. In particular, the following **afs** (where α, α_1, α_2 denote atomic **rfs**) are theorems of AICP (hence, p-valid **afs** of $\mathscr{L}^P$ because of Proposition 5.2.1) and restate known intuitionistic theorems (note that the p-validity of the first and second **afs** has already been recognized in Section 4 without making reference to AIPC).

Weak double negation law: $(\vdash\alpha)C(N(N\vdash\alpha))$.

Brouwer Law: $(N(N(N\vdash\alpha)))E(N\vdash\alpha)$.

Weak counternominal law: $((\vdash\alpha_1)C(\vdash\alpha_2))C((N\vdash\alpha_2)C(N\vdash\alpha_1))$.

On the contrary, the following **afs** (the first of which has already been considered in Section 4) do not correspond to intuitionistic theorems and are not pragmatically valid.

Strong double negation law: $((\vdash\alpha)E(N(N\vdash\alpha)))$.

Excluded middle: $((\vdash\alpha)A(N(\vdash\alpha))$.

Strong counternominal law: $((\vdash\alpha_1)C(\vdash\alpha_2))E(N\vdash\alpha_2)C(N\vdash\alpha_1))$.

(ii) We recall from Remark 3.3 (ii) that two binary relations can be introduced in $\mathscr{L}^P$, the relation of *logical implication*, defined on ψ_R, and the relation of *logical inference*, defined on ψ_A. Then, let us consider our translations of classical and intuitionistic propositional calculus into $\mathscr{L}^P$, that lead to the calculi ACPC and AIPC respectively. The axioms of ACPC are elementary **afs** of the form $\vdash(\alpha_1 \to \alpha_2)$ (with $\alpha_1, \alpha_2 \in \psi_R$) which are p-valid, so that $\alpha_1 \to \alpha_2$ is a tautology (Proposition 3.2, PV_1) and α_1 is in the relation of logical implication with α_2 (hence $\vdash\alpha_1$ is in the relation of logical inference with $\vdash\alpha_2$); it follows from the axioms of ACPC that at least one of the **rfs** α_1, α_2 is a molecular **rf**. The axioms of AIPC are complex **afs** of the form $\delta_1 C \delta_2$ (with $\delta_1, \delta_2 \in \psi_A^I$) which are p-valid, so that δ_1 is in the relation of logical inference with δ_2; in addition, δ_1 and δ_2 actually belong to ψ_A^I, i.e. only atomic radicals (propositional letters) appear in them. Therefore, we can maintain that ACPC and AIPC deal with different subjects, and deny that they can be considered as alternative theories of the same subject; in other words, we can say that ACPC and AIPC axiomatically state properties of different metalinguistic concepts, that we can identify with truth and constructive proof respectively. Thus, we recover a known thesis by Kneale and Kneale (1962), who interpret Intuitionistic Logic as an axiomatic theory of the constructive proof concept rather than an alternative to Classical Logic.[1]

It must be noted that the constructive character of Intuitionistic Logic is formalized in our approach by imposing a syntactic restriction, that is, the request that only atomic **rfs** appear in the intuitionistic **afs** (hence an intuitionistic elementary **af** admits only empirical proofs). This condition on the syntactic form of the intuitionistic formulas is necessary and sufficient in order to capture in $\mathscr{L}^P$ the notion of *direct*, or *canonical*, proof (see Prawitz, 1977, 1987), which selects those, and only those, constructive demonstrations that are intuitionistically valid. This notwithstanding, we need not make reference to a non classical metalinguistic notion of proof for intuitionistic logic in our present context: indeed, recalling Criteria 3.1, 3.2, and Remark 3.1, (i), we see that our pragmatic interpretation of IPC recovers intuitionistic logic within a classical metalanguage.

(iii) It is apparent that no p-valid **af** is contained in the set $\psi_A^e \cap \psi_A^I$ of all elementary intuitionistic **afs** (which contains elementary **afs** with atomic radical part only). Hence the set $\mathscr{V}$ of all p-valid **afs** of $\mathscr{L}^P$can be partitioned as follows,

$$\mathscr{V} = \mathscr{V}_C \cup \mathscr{V}_I \cup \mathscr{V}_P,$$

where $\mathscr{V}_C$ is the set of all p-valid **afs** in ψ_A^e (that is, the set of all theorems of ACPC, as Proposition 5.1.1 shows), $\mathscr{V}_I$ is the set of all p-valid **afs** in ψ_A^I (that is, the set of all theorems of AIPC, as Proposition 5.2.1 shows), and $\mathscr{V}_P = \mathscr{V} \backslash (\mathscr{V}_C \cup \mathscr{V}_I)$.

It follows from our treatment in Sections 3–5 that every **af** in $\mathscr{V}_C$ is an elementary **af** with a molecular radical (tautology), every **af** in $\mathscr{V}_I$ is a complex **af** that contains only atomic radicals. Therefore, the set $\mathscr{V}_P$ consists of complex **afs** which contain at least one molecular **rf**. Whenever an **af** $\delta \in \mathscr{V}_P$ takes the form $\delta = \delta_1 C \delta_2$, or $\delta = \delta_2 C \delta_1$, or $\delta = \delta_1 E \delta_2$, with

$\delta_1 \in \psi_A^e$ and $\delta_2 \in \psi_A \setminus \psi_A^e$, it establishes an explicit relationship between semantic and pragmatic connectives (a *bridge-principle*). We have already seen some examples of p-valid **afs** of this kind (see formulas (i)–(v) in Section 4). Furthermore, whenever an **af** $\delta \in \mathscr{V}_P$ takes the form $\delta = \delta_1 C \delta_2$, or $\delta = \delta_2 C \delta_1$, or $\delta = \delta_1 E \delta_2$, with $\delta_1 \in \mathscr{V}_C$ and $\delta_2 \in \mathscr{V}_I$, it establishes a relationship between ACPC and AIPC theorems.

6. CONCLUDING REMARKS

We would like to conclude our paper with some comments which show that our classical pragmatically extended language $\mathscr{L}^P$ is adequate to the goals that have led to introduce it.

As we have specified in the Introduction, the first of our goals was to prove that the conflict between correspondentist (classical) and verificationist (intuitionistic) theories of truth (and meaning) can be settled by suitably integrating these perspectives. Therefore, let us recall that many authors have provided arguments against the verificationist theories of truth (see Strawson, 1976–77; Peacocke, 1981; McDowell, 1981; Appiah, 1986; Loar, 1987). Without entering in detail, we can say that the basic problematic aspect of verificationist theories, as discussed by Dummett (1975, 1976, 1978, 1980, 1982), Prawitz (1980, 1987) and, with some variations, by Putnam (1978, 1979, 1980), is the assumption that a proposition is true iff asserting it is justified,[3] which leads to identify the notion of *truth* and the notion of *justification* (or *proof*, or *demonstration*, or *verification*). Indeed, one can argue that this viewpoint confuses a semantic notion (that is, the meaning of the word "truth") with a pragmatic criterion, whose aim is to provide a method in order to establish whether a sentence (or a proposition) is true or false (see Haak 1978, VII, I). This objection has been maintained, in particular, by Russell (1940, Chapters 20–23; 1950), Carnap (1949), Popper (1969, Introduction and Chapter 10), who have pointed out that the identification of truth and *truth criterion* (or justification) is a relevant source of philosophical misunderstandings.

Two intuitive arguments at least strongly support the need of avoiding the identification between truth and justification. The first argument was proposed by Russell (1940, Chapter 20), Carnap (1966, Chapter XXI) and Popper (1969, Introduction and Chapter 10) by observing that the pragmatic notion of justification (or proof) presupposes the semantic notion of truth as a regulative concept, since, intuitively, a proof of a proposition amounts to a proof that its truth value is "true". The second argument can be synthetized by saying that it seems reasonable to require that a sentence can be true (false) independently of our ability to recognize it as such, since, according to the classical conception of truth and meaning, there are truths (falsities), both factual and logico-mathematical, which are *undecidable*, that is epistemically inaccessible to us (see Bradley and

Swartz, 1979, pp. 167–168 and 172–174). Indeed, classically, a sentence has meaning (hence a truth value) iff: (i) it is syntactically correct (well formed); (ii) every expression which occurs in it is interpreted, that is, it is endowed with meaning (see Carnap, 1932; Russell, 1940, Chapter 17). Hence, it may happen that a given sentence satisfies both conditions (i) and (ii) (hence it is true or false) and that, nevertheless, we cannot prove neither its truth nor the truth of its negation.

In our approach the above objections have been taken into account. Indeed, we preserve the semantic (Tarskian) notion of truth (assumed here, following Tarski, to coincide with the classical notion of truth as correspondence), integrating it with the pragmatic notion of justification, interpreted as a distinct truth criterion, and not as an alternative notion of truth. Thus, in our context *truth* and *justification* belong to different semiotic fields, are endowed with different syntactic counterparts and are integrated in a wider logical-semiotic perspective. More specifically, the notion of *truth* is defined in classical Tarskian terms in the semantic of $\mathscr{L}^P$ and applies to the radical wffs of $\mathscr{L}^P$ only (Subsection 2.2), while the notion of *justification* is defined in terms of *proof* in the pragmatics of $\mathscr{L}^P$ and applies to the assertive wffs of $\mathscr{L}^P$ only, in such a way that the justification value of an assertive formula depends on the truth values of its radical subformulas (Subsection 2.3); in particular, rule JR_1 in Definition 2.3.1 defines the *justification* of an assertive elementary formula $\vdash\alpha$ of $\mathscr{L}^P$ in terms of the *proof of the truth* of its radical subformula α, consistently with the argument above against the identification of *justification* and *truth*.[4]

In addition, we note that the notions of decidability and undecidability can be suitably formalized in our context by assuming that, whenever an assignment function $\sigma \in \Sigma$ and a pragmatic evaluation function $\pi_\sigma \in \Pi_\sigma$ are given, a radical formula α is *decidable* (*undecidable*) iff the assertive formula $(\vdash\alpha)A(\vdash\neg\alpha)$ is *justified* (*unjustified*). Indeed, whenever this formalization is accepted, α is decidable for given σ and π_σ iff either a proof exists of the truth of α or a proof exists of the falsity of α, i.e., iff either $\vdash\alpha$ or $\vdash\neg\alpha$ is justified, which is consistent with the intuitive notion of decidability and reduces it to the notion of justification. Furthermore, one can introduce a second level decidability, as follows: whenever an assignment function σ and a pragmatic evaluation function π_σ are given, we say that a radical formula α is decidable (undecidable) at the second level iff the assertive formula $(\vdash\alpha)A(N\vdash\alpha)$ is justified (unjustified). This definition intuitively means that a sentence is decidable (undecidable) at the second level, for given σ and π_σ, whenever it is possible (impossible) to decide whether it is decidable, and proves that also the notion of decidability at the second level can be reduced to the notion of justification in our approach.[5]

It is important to observe that the notions of *truth* and *justification* defined in the semantics and pragmatics of $\mathscr{L}^P$, respectively, also exhibit a different logical behavior (see Remarks 2.3.1). In fact, the semantic assignment function σ satisfies the truth rules TR_1–TR_2 (Subsection 2.2) which define the meaning of the semantic connectives by assuming that they conform to classical logical laws; the pragmatic assignment function π_σ satisfies the justification rules JR_1–JR_3 (Subsection 2.3) which define the meaning of the pragmatic connectives by assuming that they conform to intuitionistic-like logical laws. Thus $\mathscr{L}^P$ explicitly shows that Dummett's fundamental thesis, which retains that shifting from a correspondence (realistic) theory of truth and meaning to a verificationist (antirealistic) theory involves a revision of logic in an intuitionistic frame, can be reinterpreted in an integrated perspective.

All the above arguments support the integration of the notions of *truth* and *justification* established in $\mathscr{L}^P$. This integration is the basic tool in order to attain our second main goal in this paper, i.e., the settlement of the conflict between classical and intuitionistic logic in a unified perspective which allows us to maintain the principle of globality (or universality) of logic as a fundamental criterion of rationality. This settlement is philosophically important; indeed, if classical and intuitionistic logic are thought of as alternative logics, they cannot be both correct, since they are not compatible. Therefore, the acceptance of intuitionistic logic would imply, as intuitionists argue, the refusal of classical logic. But it is rather difficult to imagine how we can give up classical logic and go on reasoning (see Kneale and Kneale, 1962, IX, 5). Thus, if we neither want to give up intuitionistic logic (and we have plenty of reasons for not doing this) we need to contrive a way for making it compatible with classical logic.

Before discussing our attempt of giving an adequate solution to this problem by translating the classical propositional calculus (CPC) and the intuitionistic propositional calculus (IPC) into $\mathscr{L}^P$, we recall that some attempts of recovering the compatibility of classical logic and intuitionistic logic have been done by introducing a "localist" conception of logic (see Dalla Chiara 1974, Chapter 6). This makes the correctness of the logic depend on the point of view or on the theoretical context. Nevertheless, these attempts have the unpleasant consequence of allowing logic to change with the field of research or with the theory: each theory can be endowed with its own specific logic. This implies that we are compelled to deny the universality of logic and are left without an important criterion of rational evaluation (see Haak 1978 Chapter 12; Garola, 1992c). Moreover these attempts, as Prawitz (1980) observed, are actually unsuitable to settle the conflict between the two logics.

Let us consider now the translations of CPC and IPC into $\mathscr{L}^P$ introduced in Subsections 5.1 and 5.2, which map the former onto ACPC and the latter onto AIPC. These translations are compatible, since ACPC and AIPC are constructed on different subsets of (p-valid) formulas of $\mathscr{L}^P$. Indeed, the set of theorems of ACPC is the set $\mathscr{V}_C$ of all p-valid assertive elementary formulas with tautological **rfs**, while the set of theorems of

AIPC is the set $\mathcal{V}_I$ of all p-valid complex assertive formulas with atomic radicals (see Remark 5.2.1 (iii); the restriction to atomic radicals is required by the constructive character of intuitionistic logic, see Remark 5.2.1 (ii)). Therefore, ACPC and AIPC integrate each other, unlike CPC and IPC, which are alternative incompatible logical systems. Thus, it becomes possible to assert the compatibility of classical and intuitionistic calculus, and we are able to recover the universality of logic in a frame of *global pluralism* (see the Introduction), as desired.

We notice that our translations of CPC and IPC into $\mathcal{L}^P$ conform to Quine's thesis (1970, Chapter 6) that *change of logic*, *change of subject*, which is coherent with the point of view of global pluralism (Haak, 1978, Chapter 12). In fact, the logical vocabulary of $\mathcal{L}^P$ is wider than the classical one since it also includes pragmatic connectives besides semantic ones. The axioms of ACPC state formal properties of the semantic connectives, while the axioms of AIPC state formal properties of the pragmatic connectives. The connectives of the two calculi are interpreted in semiotically different ways, so that we can assert that we have settled the conflict between the two logical systems even from this viewpoint.

We would like to underline that the pragmatic translation of IPC in $\mathcal{L}^P$ does not imply any linguistic ascent, which occurs, on the contrary, in the modal translation of IPC (Remark 3.1 (i)). It follows that our pragmatic interpretation of intuitionistic logic in terms of *justification* more strictly complies with the standard interpretation and also permits to enlight the "mysteries of the intuitionistic truth" (Van Dalen, 1986).

Our task is thus completed. Let us close our work with some remarks on possible further developments of the perspective propounded here.

It is apparent that the language $\mathcal{L}^P$ could be further enlarged in two ways. First, the apparatus of the logical-semantic signs and of the descriptive signs belonging to the vocabulary $\mathcal{A}^P$ of $\mathcal{L}^P$ could be enriched by introducing quantifiers, alethic and epistemic modal operators, individual and predicative variables and constants (we have commented in Remark 2.1.1 on the syntactic differences between force signs and modal operators); therefore, the apparatus of radical formulas of $\mathcal{L}^P$ could be extended in such a way that also the classical and the intuitionistic predicate calculi, the alethic and the epistemic modal logic can be embodied into the pragmatic language. Second, the apparatus of the logical-pragmatic signs of $\mathcal{A}^P$ could be enriched by introducing, besides the assertion sign, further signs of pragmatic mode, such as question signs, command signs, deontic modality; consequently the apparatus of the pragmatic formulas could be extended in such a way that also erotetic logic, imperative logic and deontic logic (understood as the logic of *normative*, or *prescriptive*, *sentences* and not as the logic of *norm-descriptive sentences*, or *normative propositions*) can be embodied into the pragmatic language on an *intuitionistic ground* (induced from the interpretation of pragmatic connectives). This latter extension of $\mathcal{L}^P$ would tackle the construction of the *illocutionary logic* suggested by Searle and Vanderveken (1985), as we have already observed in the Introduction, but this would be made on an intuitionistic basis, and in the framework of the integrationist logical-semiotic perspective of Morris and Carnap.[4]

The above possible enlargements of $\mathscr{L}^P$ show that $\mathscr{L}^P$ can be considered as a first step toward a unifying frame of great expressive power, suitable for integrating the most important logical systems in a globalist perspective.

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NOTES

* This paper is an enlarged and entirely revised version of the paper by Dalla Pozza (1991) worked out in the framework of C.N.R. project n. 89.02281.08, and published in Italian. The basic ideas in it have been propounded since 1986 by Dalla Pozza in a series of seminars given at the University of Lecce and in other Italian Universities. C. Garola collected the scattered parts of the work, helped in solving some conceptual difficulties and refining the formalism, yielded the proofs of some propositions (in particular, in Section 3) and provided physical examples (see in particular Remark 2.3.1).

[1] Analogous results regarding a similar problem have been recently achieved by one of the authors (Garola, 1991, 1992a, 1992b, 1992c, 1993, 1994), who has proven in particular that Quantum Logics can be interpreted as theories of testability in Quantum Physics rather than as quantum theories of truth, alternative to Classical Logic.

[2] Our quantum example is based on the distinction between truth and (empirical) justification in the language of QP. It must however be observed that this distinction is not accepted in the standard interpretation of QP, so that our example actually refers to the interpretation proposed by one of the authors (see footnote 1), where the distinction between truth and testability is considered as a basic tool for avoiding the paradoxes that afflict the canonical approach to QP.

[3] The verificationist theory of truth is classified by Dummett as *anti-realistic*, since it rejects any notion of truth that applies to sentences which are not epistemically accessible (verifiable, assertable), opposed to the correspondence theory of truth, classified as *realistic* since it admits a notion of truth which goes beyond epistemic accessibility. Therefore, the traditional *ontological* (*metaphysical*) opposition between realism and anti-realism is reinterpreted by Dummet as a semantic contrast between correspondentism and verificationism.

However, it must be observed that, even if Dummett's reinterpretation certainly grasps an important epistemological aspect of the contrast between realism and anti-realism (see Loar, 1987), it also implies an alteration of traditional notions (see Taylor, 1987). Indeed, according to the traditional ontological viewpoint, realism (both platonic and materialistic) is the philosophical position which retains that the world consists of external states of affairs (facts) and objects which do not depend on mind, experience or observation, while anti-realism (including idealism, empirism and phenomenism) is the position which retains that the world is made up by mental, or empirical, or phenomenical states of affairs and objects. Instead, according to Dummett's semantic viewpoint, realism is identified with the position according to which sentences (or the propositions expressed by sentences) can be endowed with truth values that are not epistemically accessible, i.e., that cannot be verified or falsified; on the other hand anti-realism is identified with a position according to which the truth and the falsity of a sentence are equivalent to (or at least imply) their verifiability or falsifiability, respectively.

The traditional viewpoint and Dummet's position are not equivalent. This can be seen by observing that the theory of truth as correspondence is *ontologically neutral* from a conventional perspective, since it does not depend on our beliefs on the nature of the world (in this sense Tarski could coherently assert at the same time that his semantic theory of truth was a reconstruction of the classical theory of correspondence and that it was philosophically neutral), hence neither it is realistic nor anti-realistic, while it is not *semantically neutral*, since it admits a notion of truth (and falsity) which goes beyond verifiability (and falsifiability). But because of this latter feature it should be classified as realistic according to Dummett, who disregards the difference between ontologic and semantic neutrality. Therefore, Dummett concluded that the Tarskian theory of truth, being ontologically neutral, does not yield an explication of the classical theory of truth as correspondence, and it reduces to a formal method for constructing definitions of "true in L", where L is a formalized language.

We also observe that Dummett's attack against realism (as intended in the meaning explained above) depends on the thesis that an adequate theory of meaning should explain the speaker's understanding of the meaning. But, this claim by Dummett cancels the basic distinction between a semantic theory of meaning, which requires only the specification of the (classical) *truth conditions*, and a pragmatic theory of the understanding of meaning, which seems to require also the specification of *assertability conditions* (*or justification conditions*) (see Moriconi and Napoli, 1987). Therefore, Dummett's anti-realism seems to rest on the same overlapping of semantics and pragmatics which characterizes the identification of truth and justification in verificationist theories.

[4] We note that $\mathscr{L}^P$ incorporates the logical-semiotic integrationist perspective of Morris and Carnap. In particular, the abstract notion of *assertion*, formalized in $\mathscr{L}^P$, can be considered as a first step towards the construction, suggested by Morris (1963) and accepted by Carnap (1963), of a *pure* (*logical*, *formal*) pragmatics, connected with a *pure* syntax and a *pure* semantics, so that one can provide an extension of logic which includes pure pragmatics; the notion of logic would thus embody the whole field of pure semiotics, according to an idea originally forwarded by Peirce (see Morris, 1963).

[5] Our formalization of decidability allows us to explain in our present framework the intuitionistic refusal of the "excluded middle" law. Indeed, bearing in mind our translation of the Van Dalen IPC into AIPC in Subsection 5.2, this law would correspond to the (metalinguistic) statement that $(\vdash\alpha)A(N\vdash\alpha)$ is p-valid (see Remark 5.2.1, (i)) whenever α is an atomic radical formula. Now, α atomic implies that α admits any (empirical) interpretation, so that it may happen that no empirical proof exists that α is true and no proof exists that α cannot be proven. In this case $(\vdash\alpha)A(N\vdash\alpha)$ is not justified (Definition 2.3.1, JR_3 (ii)), hence it is not p-valid, which is intuitively obvious since its p-validity would imply the presumption that all problems are solvable (Kneale and Kneale, 1962, Chapter X, Section 3). More generally, let us consider the assertive formulas $((\vdash\alpha)A(\vdash\neg\alpha)$, $(\vdash\alpha)A(N\vdash\alpha)$, $\vdash(\alpha\vee\neg\alpha)$. Then, $(\vdash\alpha)A(\vdash\neg\alpha)$ is p-valid whenever α is a tautology or a contradiction (PV_4 and PV_1 in Proposition 3.2), while it is not p-valid whenever α is atomic; $(\vdash\alpha)A(N\vdash\alpha)$ is p-valid whenever α is a tautology or it is proven that, for every $\sigma\in\Sigma$ and $\pi_\sigma\in\Pi_\sigma$, α cannot be proven (PV_4 and PV_1 in Proposition 3.2, JR_2 in Definition 2.3.1, Definition 3.1), while it is not p-valid whenever α is atomic; $\vdash(\alpha\vee\neg\alpha)$ is p-valid since it is a tautology in classical propositional logic (PV_1 in Proposition 3.2). Now, $(\vdash\alpha)A(\vdash\neg\alpha)$, when p-valid, states the decidability of α for every $\sigma\in\Sigma$ and $\pi_\sigma\in\Pi_\sigma$; then the above result that limits its p-validity is intuitively correct, since the p-validity of $(\vdash\alpha)A(\vdash\neg\alpha)$ for every radical formula α would entail a strong presumption of omniscience. Analogously, the formula $(\vdash\alpha)A(N\vdash\alpha)$, when p-valid, states the second level decidability of α for every $\sigma\in\Sigma$ and $\pi_\sigma\in\Pi_\sigma$; then, the above result that limits its p-validity is also intuitively correct, since the

p-validity of $(\vdash\alpha)A(N\vdash\alpha)$ for every radical formula α would entail a weakened presumption of omniscience. On the contrary, the p-validity of $\vdash(\alpha \vee \neg\alpha)$ does not imply any kind of decidability of α, hence it does not entail a (strong or weak) presumption of omniscience. It follows, in particular, that the metaphoric interpretation of CL as being a "God's logic" (Dalla Chiara, 1974, Sections 3–5) does not seem appropriate in our context.

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